Journal of Applied Logic 9 (2011) 95–112

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Norms as ascriptions of violations: An analysis in modal logic

Davide Grossi

Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090GE, Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Available online 8 April 2010

Keywords:
Modal logic
Counts-as
Ascription
Anderson's reduction
Chisholm's paradox
Jørgensen's dilemma

ABSTRACT

The paper proposes a formal analysis of a theory of norms resulting from pulling together Anderson's reduction, the analysis of counts-as, and a novel approach to the formal representation of language granularity in modal logic. We refer to such theory as the ascriptive view of norms. Concretely, the paper proposes a new formal definition of counts-as statements which is used as a basis for a new reduction of the deontic notion of obligation. The formal properties of these new notions are thoroughly investigated and put in perspective with related work. Finally, they are also applied to provide novel formal analyses of standard benchmark problems in deontic logic such as Chisholm's paradox and Jørgensen's dilemma.

1. Introduction

The present paper pulls together independent research threads which have been pursued in the literature on the (formal) analysis of norms. Such threads are the reductionist approach to norms started with [1–3,26], the study of counts-as initiated in [37,38] and first pursued with formal means in [24], and an ascriptive view of norms which we can track back to [34] and which has been developed in legal theory, among others, in [23]. According to this latter, norms are actually ascriptions of deontic properties to actions or states of affairs. In short, to state norms means to create new properties, which are somehow inexistent in reality (e.g., Anderson's "violation"), to create new words to name them, and consequently to predicate them of the relevant states of affairs or actions.

The paper proposes a formal analysis of this view of norms which builds, in the first place, on Anderson's reduction, in the second place, on the formal analysis of counts-as developed in [20,19,18,21,15] and, in the third place, on a formal characterization of the language creation aspect of the ascriptive view of norms. As a result, a comprehensive formal theory of norms is presented and formalized in modal logic.

The paper is structured as follows. Section 2 summarizes Anderson's reduction and provides a contextual version of it. Part of the section consists of a summary of the results presented in [20,19,18,21,15] and provides the ground for a counts-as based view of Anderson's reduction. At the end of the section the ascriptive view of norms is exposed in more details to introduce Section 3. There, a language-based notion of indistinguishability between propositional models is introduced and a modal logic, first studied in [28,27], is exposed for reasoning about it. This language-based notion of indistinguishability will be the key for capturing the phenomenon of language creation inherent in the ascriptive view of norms. A simple example is used throughout the exposition of the formalism. Section 4 applies the formalisms presented in Sections 2 and 3, providing a formal characterization of the ascriptive view of norms in the guise of a new notion of counts-as and a corresponding reduction of deontic operators. This new notion of counts-as and the logic of obligation resulting from the corresponding reduction are investigated from the point of view of their structural properties, compared with the forms of counts-as studied in [20,19,18,21,15], with standard deontic logic and, finally, also with Makinson's work at the intersection of classical and non-monotonic logic [32,33]. Section 5 puts the formal definitions of Section 4 at work providing a formal

E-mail address: d.grossi@uva.nl.

1570-8683/$ – see front matter
doi:10.1016/j.jal.2010.03.002

analysis of the Chisholm's paradox [11] and of Jørgensen's dilemma [25]. Finally, Section 6 draws some conclusions and sketches future research lines. Appendix A provides further formal details about Section 2.

2. Anderson's reduction revisited

The present section introduces a version of Anderson's reduction based on a modal logic of context.

2.1. Anderson's reduction: a brief historical look

By 'Anderson's reduction' the present paper intends, in general, the approach to deontic logic which interprets deontic notions (e.g., 'obligation', 'permission', 'prohibition') in terms of evaluative or prohairetic ones (e.g., 'good', 'ideal, 'bad', 'violation'). In [10] the observation was first put forth that statements such as 'it ought to be the case that φ' could be analyzed as 'either φ or otherwise something bad is the case' (more formally, 'if $\neg\varphi$ then $\mathbb{V}$'). As neatly put in [4], the formalization of such intuition via material implication, i.e., $\mathsf{O}\varphi := \neg\varphi \rightarrow \mathbb{V}$ would not do as, in propositional logic, such definition would make formula $\varphi \rightarrow \mathsf{O}\varphi$ valid.[1] So the nature of the reduction really resides in the formal rendering of the 'if ... then ...' locution in 'if $\neg\varphi$ then $\mathbb{V}$'. Anderson and, independently, Kanger have addressed this problem by modal means in [1,3,2] and, respectively, in [26].[2]

In particular, [1] shows that, by expanding the alphabet of monadic modal logic with a dedicated atom $\mathbb{V}$, standard deontic logic, i.e., modal system **D**, can be embedded as a subsystem in logic **T** extended with axiom $\neg\Box\mathbb{V}$ (stating that the violation is not 'necessary') and the following definition:

$$\mathsf{O}\varphi := (\Diamond\varphi \wedge \Diamond\neg\varphi) \wedge \neg\Diamond(\neg\varphi \wedge \neg\mathbb{V}) \tag{1}$$

that is, φ ought to be the case if and only if it is 'contingent' and it is 'impossible' that its negation occurs together with no violation, where 'contingent' and 'impossible' have to be interpreted as the diamonds of system **T**[3] or of a stronger system. As a consequence:

> "[...] any system of alethic modal logic [systems included between **T** and **S5**] (satisfying certain minimal conditions) gives rise, by addition of a propositional constant [the violation constant] and suitable definitions, to a system of deontic logic" [3, p. 100]

However, as shown in [6], the reduction can be obtained already within **K** extended with the axiom $\Diamond\neg\mathbb{V}$ and with the sort of definition that is nowadays most typically referred to as 'Anderson's reduction', that is:

$$\mathsf{O}\varphi := \Box(\neg\varphi \rightarrow \mathbb{V}) \tag{2}$$

In general, depending on the logic of the $\Box$ operator, different versions of the reduction can be obtained, and various alternative versions of Anderson's reduction have been proposed, also recently, in the literature (see, for instance, [12,28, 29]).

2.2. Terminological necessities

We start off by presenting the form of reduction based on system **S5**, which was already taken into consideration in [3], and to which we add a semantic twist. It is well-known that **S5** is the modal logic of universal quantification since the so-called universal modality (i.e., the modality interpreted on the $W \times W$, where W is the model's domain) is an **S5** modality [9]. Now, viewing the $\Box$ modality in Anderson's reduction as the universal modality, which we denote by $[u]$, conveys a key semantic hint:

$$\mathcal{M}, w \models [u](\neg\varphi \rightarrow \mathbb{V}) \quad \text{iff} \quad \forall w' \in W : \mathcal{M}, w' \models \neg\varphi \rightarrow \mathbb{V} \tag{3}$$

$$\text{iff} \quad \mathcal{I}(\neg\varphi) \subseteq \mathcal{I}(\mathbb{V}) \tag{4}$$

where $\mathcal{M}$ is a model for the modal language with universal modality $[u]$, W is its domain and $\mathcal{I}$ its valuation function. Formulae (3) and (4) show a very precise interpretation of Anderson's reduction: φ is obligatory means that all states (i.e., possible worlds) are such that either φ is true or, if φ is false, then a violation is also true. In this view, deontic statements amount to set-theoretic relations concerning the interpretation $\mathcal{I}(\mathbb{V})$ of the atom $\mathbb{V}$.[4]

[1] As Anderson himself puts it:

> "I hope we can all agree that such a supposition should really occasion nothing more than general laughter" [4, p. 349].

[2] It might be instructive to recall that Kanger's reduction makes use of a constant $\mathbb{Q}$ denoting normative ideality, or the absence of violation [26]. In this case, the fact that φ is obligatory means that 'if $\mathbb{Q}$ then φ'. From a formal point of view this amounts to a contrapositive version of Anderson's reduction.

[3] The proof was done syntactically by deriving the axioms of **D** from $\mathbf{T} \cup \{\neg\Box\mathbb{V}, \text{Formula (1)}\}$.

[4] For more details on this view, the reader is referred to [15].

If the deontic statements of a normative system can be represented by modal formulae involving the universal modality and the violation atom, what happens if we want to consider, under the same formalism, deontic statements belonging to several different normative systems? Technically speaking, we then look for operators that can "locally" behave like a universal modality, but that can "globally" behave in a weaker way allowing for the representation of different and possibly inconsistent deontic statements at the same time. We should find a multi-modal logic such that: a) the logic enables as many modalities as the normative systems we intend to represent; b) these modalities retain as many characteristics of $[u]$ as possible; c) the logic allows for the satisfiability of expressions such as: $[i](\neg\varphi \to \mathbb{V}) \wedge \neg[j](\neg\varphi \to \mathbb{V})$. To put it roughly, we look for a modal logic by means of which to express *contextual* terminological necessity.

2.3. A modal logic of context

In logic, contexts have been studied as sets of models [14]. Now, if the models considered are models of propositional languages, then contexts can be studied as sets of possible worlds [39]. The present section exposes a logic based on this view.[5] This logic will then be used to characterize the type of necessity involved in Anderson's reduction, thereby delivering a contextual version of it.

2.3.1. Syntax of $\mathbf{Cxt^u}$

The syntax of $\mathbf{Cxt^u}$ is the syntax of a multi-modal language $\mathcal{L}_n^{Cxt}$ [9] where n is the cardinality of the set C of context indexes. The alphabet of $\mathcal{L}_n^{Cxt}$ contains: an at most countable set $\mathbf{P}$ of propositional atoms p; the set of boolean connectives $\{\neg, \wedge\}$; a finite non-empty set C of context indexes containing the constant index u of the universal context. Metavariables $i, j, \ldots$ are used to denote elements of C. The set of formulae φ of $\mathcal{L}_n^{Cxt}$ is defined by the following BNF:

$$\mathcal{L}_n^{Cxt} : \varphi ::= \top | p | \neg\varphi | \varphi \wedge \varphi | [i]\varphi$$

where i denotes elements in C. The remaining Boolean connectives $\vee$ and $\to$ as well as the dual modal operator $\langle i \rangle$ can be defined as usual.

2.3.2. Semantics of $\mathbf{Cxt^u}$

Language $\mathcal{L}_n^{Cxt}$ is given a semantics via the class of Cxt frames $\mathcal{F} = \langle W, \{W_i\}_{i\in\mathrm{C}}\rangle$ such that:

- W is a non-empty set of states;
- $\{W_i\}_{i\in\mathrm{C}}$ is a family of subsets of W such that $W_u = W \in \{W_i\}_{i\in\mathrm{C}}$.

Intuitively, these frames consist of the domain W and of a finite number $n = |\mathrm{C}|$ of subsets of W among which W itself. Such subsets straightforwardly model the conception of contexts as sets of (propositional) models sketched above. Notice that the domain W represents the global, or universal, context.

Models are, as usual, structures $\langle \mathcal{F}, \mathcal{I}\rangle$ where $\mathcal{F}$ belongs to the class Cxt and $\mathcal{I}$ is a valuation function $\mathcal{I} : \mathbf{P} \longrightarrow \mathcal{P}(W)$. Satisfaction is defined as follows.

Definition 1 *(Satisfaction based on* Cxt *frames).* Let $\mathcal{M}$ be a model built on a Cxt frame.

$$\mathcal{M}, w \models \top$$
$$\mathcal{M}, w \models p \quad \text{iff} \quad w \in \mathcal{I}(p)$$
$$\mathcal{M}, w \models \neg\varphi_1 \quad \text{iff} \quad \text{NOT } \mathcal{M}, w \models \varphi_1$$
$$\mathcal{M}, w \models \varphi_1 \wedge \varphi_2 \quad \text{iff} \quad \mathcal{M}, w \models \varphi_1 \text{ AND } \mathcal{M},\ w \models \varphi_2$$
$$\mathcal{M}, w \models [u]\varphi \quad \text{iff} \quad \forall w' \in W : \mathcal{M},\ w' \models \varphi$$
$$\mathcal{M}, w \models [i]\varphi \quad \text{iff} \quad \forall w' \in W_i : \mathcal{M},\ w' \models \varphi$$

where i ranges on C and $W_u = W$. As usual, we say that φ is valid in a model $\mathcal{M}$ (in symbols, $\mathcal{M} \models \varphi$) if and only if for all $w \in W$ we have that $\mathcal{M}, w \models \varphi$. Similarly, we say that φ is valid in a class of frames, e.g. Cxt (in symbols, $\mathrm{Cxt} \models \varphi$), if and only if it is valid in all the models built on the frames in the class.

Notice that the $[u]$ is the universal operator. Notice also that, while in standard modal logic the truth of $[i]$ and $\langle i \rangle$ formulae depends on the evaluation state, the truth of such formulae interpreted on Cxt frames does not. In Cxt frames truth implies validity, and this is what we would intuitively expect to be the case for the contexts of normative systems. What holds in the context of a given normative system is not determined by the point of evaluation but just by the system as such, i.e., by its own rules.

[5] Readers are again referred to [15] for a more detailed exposition.

It is instructive to notice that, technically speaking, Cxt frames are multi-sets and not Kripke frames. However they are equivalent to the class TE^{eq} of Kripke frames whose accessibility relations satisfy the following properties: i–j transitivity (if wR_iw' and $w'R_jw''$ then wR_jw''), i–j euclideanicity (if wR_iw' and wR_jw'' then $w'R_jw''$), and are such that they contain an equivalence relation R_u such that for all $i \in \mathrm{C}$, $R_i \subseteq R_u$. The interested reader can find a proof of this equivalence in Appendix A.

2.3.3. Axiomatics of $\mathbf{Cxt^u}$

Logic $\mathbf{Cxt^u}$ results from the union of the schemata of modal logic $\mathbf{K45^{ij}_n}$, which axiomatizes contexts [15], with the schemata of logic **S5** axiomatizing the behavior of the global context u, plus the interaction axiom `Incl`, which just states that u is the biggest context.

$$
\begin{array}{ll}
(\mathrm{P}) & \text{all tautologies of propositional calculus}\\
(\mathrm{K}^i) & [i](\varphi_1 \to \varphi_2) \to ([i]\varphi_1 \to [i]\varphi_2)\\
(4^{ij}) & [i]\varphi \to [j][i]\varphi\\
(5^{ij}) & \neg[i]\varphi \to [j]\neg[i]\varphi\\
(\mathrm{T}^u) & [u]\varphi \to \varphi\\
(\mathrm{Incl}) & [u]\varphi \to [i]\varphi\\
(\mathrm{Dual}) & \langle i\rangle\varphi \leftrightarrow \neg[i]\neg\varphi\\
(\mathrm{MP}) & \text{IF} \vdash \varphi_1 \text{ AND } \vdash \varphi_1 \to \varphi_2 \text{ THEN } \vdash \varphi_2\\
(\mathrm{N}^i) & \text{IF } \vdash \varphi \text{ THEN } \vdash [i]\varphi
\end{array}
$$

where i, j range over the set of indexes C and u denotes the universal context index in C. The interaction axiom `Incl` states something quite intuitive concerning the interaction of the $[u]$ operator with all other context operators: what holds in the global context, holds in every context.[6] Soundness and completeness of this axiomatization with respect to Cxt frames are proven in [15].

2.4. Anderson's reduction contextualized

Everything has been put into place to provide a contextualization of the version of Anderson's reduction sketched in Section 2.2. The fact that φ is ideally the case in context i can be formalized as $[i](\neg\varphi \to \mathrm{V})$ and read as: *the negation of* φ *necessarily implies a violation within context* i. It becomes thus possible to express that φ is obligatory in the context i of a given normative system, while $\neg\varphi$ is permitted in the context j of a different normative system: $[i](\neg\varphi \to \mathrm{V}) \wedge \langle j\rangle(\neg\varphi \wedge \neg\mathrm{V})$.

In [15] such reduction has been called a "counts-as reduction of deontic logic", and it has been proposed also in [30]. Counts-as is the locution introduced in [37,38], and formally investigated for the first time in [24], by means of which Searle presents the basic syntax of constitutive rules, that is, of the building blocks of social reality. From a semantic point of view, such locution can acquire several different meanings, some of which have been systematically analyzed in [20,19,18,21,15]. One of these senses—the classificatory counts-as—is there formalized as the strict implication in $\mathbf{Cxt^u}$:

$$\varphi_1 \Rightarrow^{cl}_i \varphi_2 := [i](\varphi_1 \to \varphi_2) \tag{5}$$

Such formalization provides the ground for the counts-as reduction of obligations as statements "φ ought to be the case in context i" as:

$$\mathsf{O}_i\varphi := \neg\varphi \Rightarrow^{cl}_i \mathrm{V} \tag{6}$$

Intuitively, the negation of φ counts as a violation in context i, meaning that the negation of φ is classified as a violation in context i.

Such reduction can be straightforwardly strengthened by considering stronger senses of counts-as. One of these is the proper classificatory counts-as, also formalizable in $\mathbf{Cxt^u}$ as:

$$\varphi_1 \Rightarrow^{cl+}_i \varphi_2 := [i](\varphi_1 \to \varphi_2) \wedge \neg[u](\varphi_1 \to \varphi_2) \tag{7}$$

---

6 It is worth noticing that, although perspicuous, schema `Incl` is not necessary in the axiomatics, since it can be derived from 4^{ij} and T^u:

$$(4^{ij}) \vdash [u]\varphi \to [i][u]\varphi$$

$$(T^u) + (\mathrm{N}) + (\mathrm{P}) + (\mathrm{MP}) \vdash [u]\varphi \to [i]\varphi$$

We are indebted to Guillaume Aucher for this observation.

which enables the following reduction:

$$\mathsf{O}_i\varphi := \neg\varphi \Rightarrow_i^{cl+} \mathtt{V}. \tag{8}$$

Intuitively, the negation of φ counts as a violation in context i, meaning that the negation of φ is classified as a violation in context i (first conjunct of the right-hand side of Formula (7)), but the negation of φ is not always classified as a violation (second conjunct of the right-hand side of Formula (7)).

2.5. Norms as ascriptions

The reduction of deontic to counts-as statements of the type displayed in Formula (7) stresses that a state of affairs properly determines a violation only within a context, since outside the context that would not necessarily be the case. In [18] and [15], the rationale behind this formal characterization was taken from Searle's words themselves:

> "[...] where the rule (or system of rules) is constitutive, behavior which is in accordance with the rule can receive specifications or descriptions which it could not receive if the rule did not exist" [37, p. 35].

Constitutive rules add something to what is already the case and proper contextual classification is a way to capture this intuition. However, there is also another way to look at the novelty introduced by constitutive rules. In a sense, what they do is to literally introduce new concepts, rather than just validating classifications which would otherwise not be valid. They create new terms to be used for a further conceptualization of reality. Such view of rules as ascriptions has a long history, starting with Pufendorf's notion of "impositio" [34, pp. 100–101] and has been proposed in legal theory, for instance, in [23]. As a matter of fact, Searle's thesis according to which institutional facts are construed upon brute ones [38] can be viewed an instance of this ascriptive view of social reality.

Now, the central aspect of ascription is language creation. In order for an ascription to take place, a new term needs to be created, which can then be used for denoting the desired property. If we take an ascriptive view of Anderson's reduction, this means that the term "violation" is introduced in order to separate desired or ideal actions or states of affairs from their undesired or sub-ideal counterparts. Interestingly enough, this exact view is neatly formulated in Jørgensen's paper which introduced his dilemma [25]:

> "How is a sentence of the form "Such and such is to be so and so" to be verified? How is it for instance to be verified that all promises are to be kept? To this question I know of no other answer than the following: The phrase "is to be etc." describes not a property which an action or a state of affairs either has or not, but a kind of quasi-property which *is ascribed* to an action or a state of affairs when a person is willing or commanding the action to be performed, resp. the state of affairs to be produced" [25, pp. 292–293]

The following sections develop a formal analysis of this ascriptive view of norms. The primary technical difficulty resides in providing a suitable formal ground for the representation of language creation. From a propositional point of view, language creation means that new propositional atoms are somehow introduced in the language and consequently evaluated in the models. Therefore, in order to model language creation in logic, we should first be able to model, within the same logical framework, different languages. This is an aspect which, at first, might look hard to capture in a standard logical framework since valuation functions are typically not partial, i.e., they evaluate all the atoms in the language. We will see that such difficulty can be overcome by an appropriate use of rather standard semantic means.

3. "In the beginning was the Word"

The present section shows how modal logic offers an elegant way to represent different languages within one same formalism, without resorting to non-standard tools such as partial valuation functions.

3.1. Adam & Eve

Consider the propositional language $\mathcal{L}$ built from the alphabet $\mathbf{P}$ of propositional atoms: $\mathtt{eat\_apple}$ ("the apple has been eaten"), $\mathtt{V}$ ("a violation has occurred"). We have of course four possible models such that: $w_1 \models \mathtt{eat\_apple} \wedge \mathtt{V}$, $w_2 \models \mathtt{eat\_apple} \wedge \neg\mathtt{V}$, $w_3 \models \neg\mathtt{eat\_apple} \wedge \mathtt{V}$ and $w_4 \models \neg\mathtt{eat\_apple} \wedge \neg\mathtt{V}$. That is, we have the state in which the apple is eaten and there is a violation (w_1), the state in which the apple is eaten but there is no violation (w_2), the state where the apple is not eaten and there is a violation (w_3), and finally the state where no apple is eaten nor there is a violation (w_4). See Fig. 1.

Obviously, all these states can be distinguished from each other. But suppose now to compare the models ignoring atom $\mathtt{V}$. Models w_1 and w_2 would not be distinguishable any more, nor would states w_3 and w_4. Which is just another way to say that, had we used a sublanguage $\mathcal{L}_i$ of $\mathcal{L}$ built on the set of atoms $\{\mathtt{eat\_apple}\}$, we would have been able

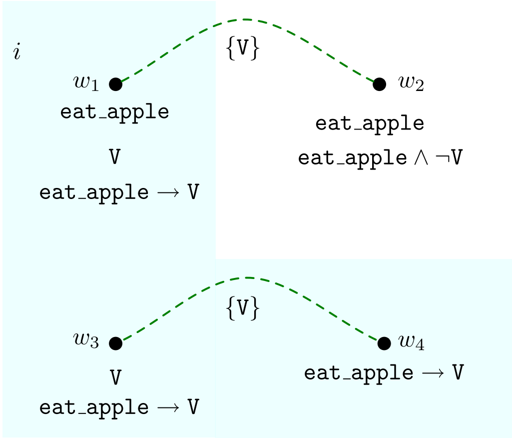

Fig. 1. $\mathsf{TE}^{eq} \otimes \mathsf{PrI}$ model of the Adam & Eve scenario.

to distinguish only states w_1 from w_3 and w_2 from w_4. Simplifying the biblical story for illustrative purposes, this latter language can be viewed as the language at disposal of Adam & Eve in their pre-moral stage, before hearing God commanding "but of the tree of the knowledge of good and evil, thou shalt not eat of it". In fact, after hearing God's command they became endowed with the possibility to discern good ($\neg\mathtt{eat\_apple}$) from evil ($\mathtt{eat\_apple}$), that is, their language was enriched and they got to distinguish also states w_1 from w_2 and w_3 from w_4. The possibility of making this distinction is inherent in fact that they are forbidden to eat the apple.

3.2. Propositional equivalence up to a signature

The intuitions sketched in the previous section are here made formal. The signature of a propositional language is its non-logical alphabet, that is, its set of propositional atoms. Let $\mathbf{P} = \{p, q, r \ldots\}$ be a countable set of propositional atoms, and let $\mathcal{L}(\mathbf{P})$ be the propositional language built on $\mathbf{P}$ and the usual Boolean connectives. We say that $\mathbf{P}$ is the signature of $\mathcal{L}(\mathbf{P})$.

Consider now the set $2^{\mathbf{P}}$ of all possible sub-signatures of $\mathcal{L}(\mathbf{P})$. Elements of such set will be denoted $P, Q, R, \ldots$, etc. Notice that the set of all sub-signatures of $\mathcal{L}(\mathbf{P})$ naturally yields a set algebra $\langle 2^{\mathbf{P}}, \cup, -, \mathbf{P}, \emptyset\rangle$. Two propositional models w and w' of $\mathcal{L}(\mathbf{P})$ are propositionally equivalent if they satisfy the same atoms in $\mathbf{P}$. As a consequence, for any formula φ of $\mathcal{L}(\mathbf{P})$: $w \models \varphi$ iff $w' \models \varphi$. If w and w' are equivalent ($w \sim w'$) then there is no set Φ of formulae of $\mathcal{L}(\mathbf{P})$ whose models contain w but not w', or vice versa. That is to say, the two models are indistinguishable for $\mathcal{L}(\mathbf{P})$.

However, two models which are not equivalent for $\mathbf{P}$ may be equivalent for some sub-signature $P \in 2^{\mathbf{P}}$. In this case, the two models cannot be distinguished by only looking at the atoms in P. The following definition makes such notion formal.

Definition 2 *(Equivalence up to a signature).* Two models w and w' for a propositional language $\mathcal{L}$ are equivalent up to signature $P \in 2^{\mathbf{P}}$, or P-equivalent, if and only if for any $p \in P$, $w \models p$ iff $w' \models p$. If w and w' are P-equivalent we write $w \sim_P w'$.

Obviously, if $w \sim_P w'$ then for all $\varphi \in \mathcal{L}(P)$: $w \models \varphi$ iff $w' \models \varphi$. The definition makes precise the idea of two propositional models agreeing up to what is expressible on a given signature.

Proposition 1 *(Properties of $\sim_P$). Let W be a set of models for the propositional language $\mathcal{L}(\mathbf{P})$. The following holds:*

(i) *For every signature $P \in 2^{\mathbf{P}}$, the relation $\sim_P$ is an equivalence relation on W;*
(ii) *For all signatures $P, Q \in 2^{\mathbf{P}}$, if $P \subseteq Q$ then $\sim_Q \subseteq \sim_P$;*

Proof. (i) The following holds: identity is a subrelation of $\sim_P$ for any sub-signature P; and that $\sim_P \circ \sim_P$ and $\sim_P^{-1}$ are subrelations of $\sim_P$ for any signature P. (ii) If $m \sim_Q m'$ then for all atoms $p \in Q$: $w \models p$ iff $w' \models p$. Therefore, since $P \subseteq Q$, $w \sim_P w'$. $\square$

Besides showing that signature-based equivalence is an equivalence relation (i), Proposition 1 shows also that: (ii) the bigger the signature, the more fine-grained is the equivalence relation.

3.3. Release logics

Propositional release logics (**PRL**) have been first introduced and studied in [28,27] in order to provide a modal logic characterization of the notion of irrelevancy. Irrelevancies are, in short, those aspects which we can choose to ignore. Irrelevancy is represented via modal release operators, specifying what is relevant to the current situation and what can instead be ignored. Release operators are indexed by an abstract 'issue' denoting what is considered to be irrelevant for evaluating the formula in the scope of the operator: $\Delta_I \varphi$ means 'formula φ holds in all states where issue I is irrelevant', or 'φ holds in all states modulo issue I' or 'φ necessarily holds while releasing issue I'. Dually, $\nabla_I \varphi$ means 'formula φ holds in at least one of the states where issue I is irrelevant', or 'φ possibly holds while releasing issue I'.

Issues can be in principle anything, but their essential feature is that they yield equivalence relations which cluster the states in the model. An issue I is conceived as something that determines a partition of the domain in clusters of states which agree on everything but I, or which are equivalent modulo I. Release operators are interpreted on these equivalence relations. As such, propositional release logic can be thought of as a "logic of controlled ignorance" [28]. They represent what we would know, and what we would ignore, by choosing to disregard some issues.

3.3.1. Syntax of **PRL**

The syntax of **PRL** is the syntax of a standard multi-modal language $\mathcal{L}_m^{Prl}$ [9] where m is the cardinality of a non-empty set $\mathtt{Iss}$ of releasable issues. The alphabet of $\mathcal{L}_m^{Prl}$ contains: an at most countable set $\mathbf{P}$ of propositional atoms p; the set of boolean connectives $\{\neg, \wedge\}$; a finite non-empty set $\mathtt{Iss}$ of issues. Metavariables $I, J, \ldots$ are used for denoting elements of $\mathtt{Iss}$. The set of formulae φ of $\mathcal{L}_n^{Prl}$ is defined by the following BNF:

$$\mathcal{L}_m^{Prl} : \varphi ::= \top | p | \neg\varphi | \varphi \wedge \varphi | \Delta_I \varphi$$

where I denotes elements in $\mathtt{Iss}$. Boolean connectives and the modal dual ∇_I can be defined as usual.

3.3.2. Structure of $\mathtt{Iss}$

One last important feature of **PRL** should be addressed before getting to the semantics. We have seen that modal operators are indexed by an issue denoting what is disregarded when evaluating the formula in the scope of the operator. The finite set $\mathtt{Iss}$ of these issues is structured as a partial order, that is to say, $\langle \mathtt{Iss}, \leqq \rangle$ is a structure on the non-empty set $\mathtt{Iss}$, where $\leqq$ ("being a sub-issue of") is a binary relation on $\mathtt{Iss}$ which is reflexive, transitive and antisymmetric. The aim of the partial order is to induce a structure on the equivalence relations denoting the release of each issue in $\mathtt{Iss}$: if $I \leqq J$ then the clusters of states obtained by releasing J contain the clusters of states obtained by releasing I. Intuitively, if I is a sub-issue of J then by disregarding J, I is also disregarded. This aspect is made explicit in the models which, for the rest, are just Kripke models.

3.3.3. Semantics of **PRL**

The semantics of **PRL** is given via the class Prl of frames $\mathcal{F} = \langle W, \{R_I\}_{I \in \mathtt{Iss}} \rangle$ such that:

- W is a non-empty set of states;
- $\{R_I\}_{I \in \mathtt{Iss}}$ is a family of equivalence relations such that: if $I \leqq J$ then $R_I \subseteq R_J$.

Models are, as usual, structures $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ where $\mathcal{I}$ is a valuation function $\mathcal{I} : \mathbf{P} \longrightarrow \mathcal{P}(W)$ associating to each atom the set of states which make it true. PRL models are therefore just $\mathbf{S5}_{|\mathtt{Iss}|}$ models with the further constraint that the granularity of the equivalence relations follows the partial order defined on the set of issues, that is, the $\leqq$-smaller is the issue released, the more granular is the partition obtained via the associated equivalence relation. The satisfaction relation is standard.

Definition 3 *(Satisfaction based on* Prl *frames).* Let $\mathcal{M}$ be model built on an PRL frame.

$$\mathcal{M}, w \models \Delta_I \varphi \quad \text{iff} \quad \forall w', w R_I w' : \mathcal{M}, w' \models \varphi$$

$$\mathcal{M}, w \models \nabla_I \varphi \quad \text{iff} \quad \exists w', w R_I w' : \mathcal{M}, w' \models \varphi$$

where $I \in \mathtt{Iss}$. Boolean clauses are omitted. As usual, a formula φ is said to be valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \models \varphi$, iff for all w in W, $\mathcal{M}, w \models \varphi$. It is said to be valid in a frame $\mathcal{F}$ (i.e., $\mathcal{F} \models \varphi$) if it is valid in all models based on that frame. Finally, it is said to be valid on the class of Prl frames (i.e., $\mathsf{Prl} \models \varphi$) if it is valid in every frame $\mathcal{F}$ in Prl.

3.3.4. Axiomatics of **PRL**

The axiomatics consists of a multi-modal **S5** plus the $\texttt{PO}$ (partial order) axiom:

$$
\begin{array}{rl}
(\texttt{P}) & \text{all tautologies of propositional calculus} \\
(\texttt{K}) & \Delta_I(\varphi_1 \to \varphi_2) \to (\Delta_I\varphi_1 \to \Delta_I\varphi_2) \\
(\texttt{T}) & \Delta_I\varphi \to \varphi \\
(4) & \Delta_I\varphi \to \Delta_I\Delta_I\varphi \\
(5) & \nabla_I\varphi \to \Delta_I\nabla_I\varphi \\
(\texttt{PO}) & \Delta_I\varphi \to \Delta_J\varphi \text{ IF } J \leqq I \\
(\texttt{Dual}) & \nabla_I\varphi \leftrightarrow \neg\Delta_I\neg\varphi \\
(\texttt{MP}) & \text{IF} \vdash \varphi_1 \text{ AND} \vdash \varphi_1 \to \varphi_2 \text{ THEN} \vdash \varphi_2 \\
(\texttt{N}^I) & \text{IF} \vdash \varphi \text{ THEN} \vdash \Delta_I\varphi
\end{array}
$$

where $I, J \in \texttt{Iss}$. A proof of the soundness and completeness of this axiomatics with respect to the class Prl frames is exposed in [27].

3.4. PRL with Boolean Algebras

In **PRL**, the partial order structure on the set $\texttt{Iss}$ is reflected by axiom $\texttt{PO}$ in the axiomatics, and by the partial order imposed on the accessibility relations in the semantics. By adding structure to the partial order on $\texttt{Iss}$ more validities can be derived which mirror that structure. Interesting for our purposes is the case when $\texttt{Iss}$ is structured according to a Boolean Algebra. The following propositions lists some of the PRL validities holding in that case.

Proposition 2 *(Validities of PRL with BA). Let* $\texttt{Iss}$ *be ordered as* $\langle \texttt{Iss}, \sqcup, \sqcap, -, 1, 0, \leqq \rangle$, *where the structure* $\langle \texttt{Iss}, \sqcup, \sqcap, -, 1, 0 \rangle$ *is a Boolean Algebra. The following formulae can be derived in PRL:*

$$\Delta_1\varphi \to \Delta_I\varphi \tag{9}$$

$$\Delta_I\varphi \to \Delta_0\varphi \tag{10}$$

$$\Delta_{I \sqcup J}\varphi \to (\Delta_I\varphi \wedge \Delta_J\varphi) \tag{11}$$

$$\Delta_I\varphi \vee \Delta_J\varphi \to \Delta_{I \sqcap J}\varphi \tag{12}$$

$$\Delta_I\varphi \leftrightarrow \Delta_{--I}\varphi \tag{13}$$

$$(\Delta_I\varphi \to \Delta_J\varphi) \leftrightarrow (\Delta_{-J}\varphi \to \Delta_{-I}\varphi) \tag{14}$$

Proof. The desired derivations are easily obtainable: some (Formulae (9), (10), (11)) are just instances of $\texttt{PO}$, some (Formulae (12), (13), (14)) can be proven by application of $\texttt{PO}$ and propositional logic. □

Notice that the "full release" operator Δ_1 expresses what necessarily holds after releasing all issues in $\texttt{Iss}$. In fact, it behaves like a universal operator (Formula (9)). Vice versa, the "empty release" operator Δ_0 expresses what necessarily holds when nothing is released. In other words, it encodes a quantification on all the worlds equivalent to the evaluation state with respect to the set of issues $\texttt{Iss}$. Formulae (11)–(14) encode rather intuitive Boolean properties. We will see, in the next section, that the specific type of release involved in the notion of ascription, enjoys precisely this structure.

3.5. Propositional sublanguage equivalence as release

Reasoning about propositional sublanguage equivalence is an instance of reasoning in release logic.

Proposition 3 *(Sublanguage equivalence models). Consider a propositional language* $\mathcal{L}$ *on the set of atoms* $\mathbf{P}$, *and a set of states* W. *Any valuation function* $\mathcal{I} : \mathbf{P} \longrightarrow \mathcal{P}(W)$ *determines an* Prl *model* $\mathcal{M} = \langle W, \{\sim_{-P}\}_{P \in 2^{\mathbf{P}}}, \mathcal{I} \rangle$ *where* $\texttt{Iss} = 2^{\mathbf{P}}$ *is ordered according to a Boolean Algebra.*

Proof. It follows from the properties of $\sim_P$ proven in Proposition 1 and from the fact that $2^{\mathbf{P}}$ naturally yields a set algebra. □

Notice that $\texttt{Iss}$ is taken to be $2^{\mathbf{P}}$, which is ordered by set-theoretic inclusion $\subseteq$. Notice also that the released issues are taken to be the complements $-P$ of each subsignature $P \in 2^{\mathbf{P}}$. In fact, what is released is just what is chosen not to be expressible. The accessibility relations are therefore taken to be the relations $\sim_{-P}$ of equivalence up to a signature.[7] So, for example Δ_0 (the empty release) is interpreted on $\sim_{\mathbf{P}}$ (standard propositional equivalence), while Δ_1 is interpreted on $\sim$ (equivalence with respect to an empty signature).

To put it roughly, what Proposition 3 says is that **PRL** is a suitable logic to reason about scenarios like the Adam & Eve one sketched in Section 3.1. Let us get back to that example. Now it is possible to represent both the pre- and post-God's commandment situations, within the same formalism, by making use of the release operators of **PRL**. Suppose Adam & Eve to be at state w_1 in the model with domain $W = \{w_1, w_2, w_3, w_4\}$ and valuation $\mathcal{I}$ as in Section 3.1. Recall that the language was built on atoms $\mathbf{P} = \{\mathtt{eat\_apple}, \mathtt{V}\}$. So let us denote with $\{\mathtt{V}\}$ and $\{\mathtt{eat\_apple}\}$ the subsignatures containing only atom $\mathtt{V}$ and, respectively, atom $\mathtt{eat\_apple}$. These subsignatures represent the releasable issues, together with the empty signature 0 and the full signature $1 = \mathbf{P}$. Let $\mathcal{M} = \langle W, \{\sim_{\{\mathtt{V}\}}, \sim_{\{\mathtt{eat\_apple}\}}, \sim_0, \sim_1\}, \mathcal{I}\rangle$ be the resulting release model. We have that:

$$\mathcal{M}, w_1 \models \mathtt{eat\_apple} \wedge \mathtt{V} \tag{15}$$

$$\mathcal{M}, w_1 \models \Delta_0(\mathtt{eat\_apple} \wedge \mathtt{V}) \tag{16}$$

$$\mathcal{M}, w_1 \models \Delta_{\{\mathtt{V}\}}\mathtt{eat\_apple} \wedge \neg\Delta_{\{\mathtt{V}\}}\mathtt{V} \tag{17}$$

So Formula (15) just states what holds in w_1, which is the actual state where Adam & Eve eat the apple committing a violation. Formula (16) does the same by saying that, if you evaluate $\mathtt{eat\_apple}$ and $\mathtt{V}$ after releasing nothing, i.e., by using the full descriptive power of the language, then both $\mathtt{eat\_apple}$ and $\mathtt{V}$ necessarily hold. In fact, in the model at issue the set of states reachable from w_1 via $\sim_{-0}$ (i.e., $\sim_1$), coincides with w_1 itself, since there are no other states in W which are equivalent with w_1 if all available atoms are used in the comparison. Hence, in the model at issue, Δ_0 refers to the current evaluation state, i.e., w_1. Formula (17) shows what are the effects of releasing atom $\mathtt{V}$. In fact, by abstracting from $\mathtt{V}$, state w_1 is not distinguishable any more from state w_2: $w_1 \sim_{-\{\mathtt{V}\}} w_2$ or, equivalently, $w_1 \sim_{\{\mathtt{eat\_apple}\}} w_2$ since $\mathbf{P} - \{\mathtt{V}\} = \{\mathtt{eat\_apple}\}$. So there exists a state $w_2 \in W$ such that $\mathcal{M}, w_2 \models \mathtt{eat\_apple} \wedge \neg\mathtt{V}$. See Fig. 1.

Formulae (16) and (17) represent Adam & Eve's situation after and, respectively, before God's commandment "but of the tree of the knowledge of good and evil, thou shalt not eat of it". Such commandment introduces a further characterization of reality, exemplified here by the notion of violation, which was not available to Adam & Eve before the commandment was uttered.

4. Modal aspects of ascriptivism

This section puts logics $\mathbf{Cxt^u}$ and **PRL** at work together. Their fusion $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ on language $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ is all we need to get the axiomatics and semantics we are interested in.[8]

The axiomatics $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ can be directly proven complete with respect to the class of frames $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ since it is well-known that fusions preserve completeness when the classes of frames characterizing the two logics involved in the fusion are closed under disjoint unions [13, Ch. 4]. With respect to this, it is important to notice that the fusion $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ considers the semantics of $\mathbf{Cxt^u}$ given in terms of TE^{eq} frames. This is necessary because Cxt frames are not closed under disjoint unions.[9] We can now proceed with our formal analysis.

4.1. Ascription formalized

Within logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ it becomes possible to define an ascriptive notion of counts-as $\varphi_1 \Rightarrow_i^{As} \varphi_2$ between any two formulae φ_1 and φ_2, where what is released in the second conjunct of the definition is the signature of the consequent φ_2. This is taken care of by the following definition.

Definition 4 *(Ascriptive counts-as: $\Rightarrow_i^{As}$).* "φ_2 is ascribed to φ_1 in context i" is formalized in the logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$, on a multi-modal language $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ containing the set of issues $\mathtt{Iss} = 2^{\mathbf{P}}$ as follows:

$$\varphi_1 \Rightarrow_i^{As} \varphi_2 := [i](\varphi_1 \to \varphi_2) \wedge \neg[i]\Delta_{\sigma(\varphi_2)}(\varphi_1 \to \varphi_2) \tag{18}$$

where function $\sigma : \mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl} \longrightarrow 2^{\mathbf{P}}$ outputs, for any formula, its signature.

---

[7] It is instructive to notice that although all models based on equivalence up to a signature are Prl models, the reverse does not hold. In fact, the axiomatization in terms of **PRL** is too weak to isolate exactly the class of Prl models. Such axiomatic characterization is not needed for the purpose of this paper. It has nonetheless been provided in [16,17], to which we refer the interested reader.

[8] Fusions are the simplest ways of combining logics. For all the technical details about fusions we refer the reader to [13, Ch. 3]. It might however be worthwhile to concisely recall the basic definitions. The fusion $\mathcal{L}' \otimes \mathcal{L}''$ of two modal languages $\mathcal{L}'$ and $\mathcal{L}''$ is $\mathcal{L}' \cup \mathcal{L}''$; the fusion of two axiom systems $\mathbf{Ax}' \otimes \mathbf{Ax}''$ is $\mathbf{Ax}' \cup \mathbf{Ax}''$; finally, the fusion of two classes of frames $\mathsf{F}' \otimes \mathsf{F}''$ is the class of frames $\langle W, R_1, \ldots, R_n, R'_1, \ldots, R'_m\rangle$ where $\langle W, R_1, \ldots, R_n\rangle$ belongs to F' and $\langle W, R'_1, \ldots, R'_m\rangle$ belongs to F''.

[9] See Appendix A.

Section 4.2 will study some properties of this counts-as operator which adds on the formal analysis of counts-as developed in [20,19,18,21,15]. First, however, notice that by setting $\varphi_2 = \mathtt{V}$ we obtain a new version of Anderson's reduction based on ascriptive counts-as.

To explore the purport of this idea for the analysis of deontic notions let us get back to our running example. God's commandment not to eat the apple is a statement $\mathtt{eat\_apple} \to \mathtt{V}$. Roughly, such statement is a stipulation specifying what God holds to be the case or, better, what holds in the context of God's commandments. Let such commandments be set Φ (possibly a singleton). Such set naturally defines a context i whose extension W_i is just the set of states satisfying Φ.[10] Since Φ contains $\mathtt{eat\_apple} \to \mathtt{V}$, such statement can be studied as a classificatory counts-as statement pertaining to the context i of divine commands. It corresponds to the validity of strict implication $[i](\mathtt{eat\_apple} \to \mathtt{V})$ in the model. To represent this, we should add contexts to the **PRL** model introduced in Section 3.5. Let it be $\mathcal{M}' = \langle W, \{W, W_i\}, \{\sim_{\{\mathtt{V}\}}, \sim_{\{\mathtt{eat\_apple}\}}, \sim_0, \sim_1\}, \mathcal{I}\rangle$. Clearly, $[i](\mathtt{eat\_apple} \to \mathtt{V})$ will be valid in $\mathcal{M}'$ if W_i does not contain state w_2, since $\mathcal{M}', w_2 \models \mathtt{eat\_apple} \wedge \neg\mathtt{V}$. Leaving technicalities aside, stating $[i](\mathtt{eat\_apple} \to \mathtt{V})$ in the Adam & Eve scenario modeled in $\mathcal{M}'$ corresponds to setting the boundaries of the context i of divine norms Φ so that the states are ruled out in which eating the apple is consistent with the non-occurrence of a violation. The scenario is depicted in Fig. 1. The context i is highlighted and the only release accessibility relations depicted are the relevant ones concerning atom $\mathtt{V}$.

We therefore get to a new form of reduction in logic $\mathbf{Cxt}^{\mathbf{u}} \otimes \mathbf{PRL}$ by instantiating Definition 4.

Definition 5 *(Obligation as ascription of violation: $\Rightarrow_i^{As} \mathtt{V}$).* "It ought to be the case that φ in context i" is formalized in the logic $\mathbf{Cxt}^{\mathbf{u}} \otimes \mathbf{PRL}$, on a multi-modal language $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ containing atom $\mathtt{V}$ and the set of issues $\mathtt{Iss} = 2^{\mathbf{P}}$, as follows:

$$\mathsf{O}_i\varphi := \neg\varphi \Rightarrow_i^{As} \mathtt{V} \tag{19}$$

Intuitively, φ ought to be the case in i if and only if $\mathtt{V}$ is ascribed to the negation of φ in context i.[11] Interestingly, Definition 5 well captures the intuition—argued upon in [4]—that the relation between $\neg\varphi$ and $\mathtt{V}$ not be of a logical kind (as in Formula (3)):

> "What we would like [...] is some sort of "if ...then—" relation which we can *stipulate* as true, as we can't for a logical or causal "if ... then—"; in the logical or causal cases the facts are already ahead of us" [4, p. 350].

The fact that $\mathtt{V}$ follows from $\neg\varphi$ is not a logical truth since, from Definition 5, we have that $\langle i\rangle\nabla_{\{\mathtt{V}\}}(\neg\varphi \wedge \neg\mathtt{V})$. In other words, the truth of $\mathsf{O}_i\varphi$ requires that the use of $\mathtt{V}$ is necessary to establish the context i of the obligation. Was $\langle i\rangle\nabla_{\{\mathtt{V}\}}(\neg\varphi \wedge \neg\mathtt{V})$ false, then the fact that $\mathtt{V}$ follows from $\neg\varphi$ would trivially hold in i. In other words, the fact that $\mathtt{V}$ follows from $\neg\varphi$ can really be viewed as a stipulation pertaining context i which ascribes $\mathtt{V}$ to $\neg\varphi$. The ascription of violation amounts to a classificatory counts-as[12] (first conjunct of the right-hand side of Formula (18)) with the further condition (second conjunct) that the implication does not hold in context i any more if it is evaluated releasing its consequent (in this case the violation atom $\mathtt{V}$). It is worth stressing that this definition pulls together several research threads: Anderson's reductionist tradition, the formal analysis of counts-as, and the notion of release.

Definition 5 represents a strengthening of Anderson's reduction along the line of Formulae (5) and (7). It is worth spending a few more words on the right-hand side of Formula (19). Its dual version better displays the key idea behind it: $\neg\langle i\rangle(\neg\varphi \wedge \neg\mathtt{V}) \wedge \langle i\rangle\nabla_{\{\mathtt{V}\}}(\neg\varphi \wedge \neg\mathtt{V})$. By releasing the consequent $\mathtt{V}$ of the ascription, it becomes impossible to distinguish states which satisfy $\mathtt{V}$ from states which falsify $\mathtt{V}$. Now, the definition says that, in order for an ascription to hold, there is a state belonging to context i from which another state w' outside of context i can be reached which is indistinguishable from w once $\mathtt{V}$ is released, and which falsifies the implicative content of the counts-as $(\neg\varphi \wedge \neg\mathtt{V})$.[13]

4.2. On the properties of ascriptive counts-as

In this section we investigate the structural properties of $\Rightarrow_i^{As}$ as they follow from Definition 4. We have the two following theorems.

Theorem 1 *(Properties of $\Rightarrow_i^{As}$: validities). Given Definition 4, the following formulae of $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ are valid in* $\mathbf{Cxt}^{\mathbf{u}} \otimes \mathbf{PRL}$:

$$\varphi_2 \leftrightarrow \varphi_3 / (\varphi_1 \Rightarrow_i^{As} \varphi_2) \leftrightarrow (\varphi_1 \Rightarrow_i^{As} \varphi_3) \tag{20}$$

$$\varphi_1 \leftrightarrow \varphi_3 / (\varphi_1 \Rightarrow_i^{As} \varphi_2) \leftrightarrow (\varphi_3 \Rightarrow_i^{As} \varphi_2) \tag{21}$$

---

[10] The definition of contexts by sets of norms has been thoroughly investigated in [21,15].

[11] Just like we have defined an obligation operator O_i we can obviously define permission and prohibition operators: $\mathsf{P}_i\varphi := \neg(\varphi \Rightarrow_i^{As} \mathtt{V})$ and, respectively, $\mathsf{F}_i\varphi := \varphi \Rightarrow_i^{As} \mathtt{V}$. Notice that, in $\mathbf{Cxt}^{\mathbf{u}} \otimes \mathbf{PRL}$, they are related in the standard way, that is: $\mathsf{F}_i\varphi$ is equivalent to $\mathsf{O}_i\neg\varphi$ which, in turn, is equivalent to $\neg\mathsf{P}_i\varphi$.

[12] Note that the stronger form of proper classificatory counts-as could also be used.

[13] Typically, state w satisfies $\neg\varphi$, that is, the antecedent of the counts-as since w and w' differ only in the interpretation of atom $\mathtt{V}$. In the Adam & Eve scenario, for instance, $w = w_1$ and $w' = w_2$.

$$\left(\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \wedge \left(\varphi_1 \Rightarrow_i^{As} \varphi_3\right)\right) \rightarrow \left(\varphi_1 \Rightarrow_i^{As} (\varphi_2 \wedge \varphi_3)\right) \tag{22}$$

$$\left(\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \wedge \left(\varphi_3 \Rightarrow_i^{As} \varphi_2\right)\right) \rightarrow \left((\varphi_1 \vee \varphi_3) \Rightarrow_i^{As} \varphi_2\right) \tag{23}$$

Proof *(Sketch).* The proof is routine. We provide the proof of the validity of Formula (22) as an example. Suppose, per absurdum, that the consequent is false. By Definition 4, there exists a model $\mathcal{M}$ and state w s.t. $\mathcal{M}, w \not\models [i](\varphi_1 \rightarrow \varphi_2 \wedge \varphi_3) \wedge \neg[i]\Delta_{\sigma(\varphi_2 \wedge \varphi_3)}(\varphi_1 \rightarrow \varphi_2 \wedge \varphi_3)$. Since from the properties of $[i]$ follows that $\mathcal{M}, w \models [i](\varphi_1 \rightarrow \varphi_2 \wedge \varphi_3)$, it also follows that $\mathcal{M}, w \models [i]\Delta_{\sigma(\varphi_2 \wedge \varphi_3)}(\varphi_1 \rightarrow \varphi_2 \wedge \varphi_3)$ which is impossible given $\mathcal{M}, w \not\models [i]\Delta_{\sigma(\varphi_2)}(\varphi_1 \rightarrow \varphi_2)$ and $\mathcal{M}, w \not\models [i]\Delta_{\sigma(\varphi_3)}(\varphi_1 \rightarrow \varphi_3)$, since $\sigma(\varphi_2) \subseteq \sigma(\varphi_2 \wedge \varphi_3)$ and $\sigma(\varphi_3) \subseteq \sigma(\varphi_2 \wedge \varphi_3)$. □

Theorem 2 *(Properties of $\Rightarrow_i^{As}$: invalidities). Given Definition 4, the following formulae of $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ are not valid in* $\mathbf{Cxt^u} \otimes \mathbf{PRL}$*:*

$$\varphi \Rightarrow_i^{As} \varphi \tag{24}$$

$$\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \rightarrow \left(\varphi_1 \wedge \varphi_3 \Rightarrow_i^{As} \varphi_2\right) \tag{25}$$

$$\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \rightarrow \left(\varphi_1 \Rightarrow_i^{As} \varphi_2 \vee \varphi_3\right) \tag{26}$$

$$\left(\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \wedge \left(\varphi_2 \Rightarrow_i^{As} \varphi_3\right)\right) \rightarrow \left(\varphi_1 \Rightarrow_i^{As} \varphi_3\right) \tag{27}$$

$$\left(\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \wedge \left(\varphi_1 \Rightarrow_i^{As} \varphi_3\right)\right) \rightarrow \left((\varphi_1 \wedge \varphi_2) \Rightarrow_i^{As} \varphi_3\right) \tag{28}$$

$$\left(\varphi_1 \Rightarrow_i^{As} \varphi_2\right) \rightarrow \left(\neg\varphi_2 \Rightarrow_i^{As} \neg\varphi_1\right) \tag{29}$$

Proof *(Sketch).* The proof is routinary. As an example, we provide countermodels for Formulae (27) and (29). Formula (27): $\forall w \in W, \mathcal{M}, w \models \varphi_1 \rightarrow \varphi_3$; $\forall w \in W_i, \mathcal{M}, w \models \varphi_1 \rightarrow \varphi_2$ and $\mathcal{M}, w \models \varphi_2 \rightarrow \varphi_3$; and $\exists w \in W_i, \exists w', w'' \in W$ s.t.: $w \sim_{-\sigma(\varphi_2)} w'$ and $\mathcal{M}, w' \models \varphi_1 \wedge \neg\varphi_2 \wedge \varphi_3$; $w \sim_{-\sigma(\varphi_3)} w''$ and $\mathcal{M}, w'' \models \neg\varphi_1 \wedge \varphi_2 \wedge \neg\varphi_3$. Formula (29): $\forall w \in W_i, \mathcal{M}, w \models \varphi_1 \rightarrow \varphi_2$ and $\exists w \in W_i, \exists w' \in W$ s.t. $w \sim_{-\sigma(\varphi_2)} w'$ and $\mathcal{M}, w' \models \varphi_1 \wedge \neg\varphi_2$ and $\forall w'' \in W$ if $w \sim_{-\sigma(\varphi_1)} w''$ then $\mathcal{M}, w'' \models \varphi_1 \rightarrow \varphi_2$. □

So, ascriptive counts-as satisfies the core of the structural properties of counts-as isolated in [24], i.e., left and right logical equivalence (Formulae (20) and (21)), disjunction of the antecedents (Formula (23)) and conjunction of the consequents (Formula (22)), and it falsifies transitivity (Formula (27)). It also falsifies reflexivity (Formula (24)), antecedent strengthening and consequent weakening (Formulae (25) and (26)), and cautious monotonicity (Formula (28)). Most interestingly, contraposition (Formula (29)) also fails. The failure of contraposition is a remarkable aspect of $\Rightarrow_i^{As}$ since contraposition was one of the problematic properties of the classificatory view of counts-as. Ascription seems therefore to be a fruitful development of the classificatory perspective pursued in the line of work presented in [20,19,18,21,15].

4.3. Ascription and other forms of counts-as

At this point it is worth spending a few words about the relative strength of the ascriptive view of counts-as with respect to the other forms of counts-as isolated in the aforementioned line of work, and in particular the classificatory counts-as ($\Rightarrow_i^{Cl}$) and the proper classificatory counts-as ($\Rightarrow_i^{Cl+}$). Let us briefly recall their definitions in logic $\mathbf{Cxt^u}$[14]:

$$\varphi_1 \Rightarrow_i^{cl} \varphi_2 := [i](\varphi_1 \rightarrow \varphi_2)$$

$$\varphi_1 \Rightarrow_i^{cl+} \varphi_2 := [i](\varphi_1 \rightarrow \varphi_2) \wedge \neg[u](\varphi_1 \rightarrow \varphi_2)$$

Obviously, for any φ_1, φ_2 it is the case that $\varphi_1 \Rightarrow_c^{cl+} \varphi_2$ strictly implies $\varphi_1 \Rightarrow_c^{cl} \varphi_2$. What about the relationship between $\Rightarrow_i^{cl+}$ and $\Rightarrow_i^{As}$? In fact, they are structurally very similar. The $\Rightarrow_i^{cl+}$-versions of Formulae (20)–(23) are all valid, and the $\Rightarrow_i^{cl+}$-versions of Formulae (24)–(28) all fail as shown in [18,15]. The only difference seems to be contraposition, which holds for $\Rightarrow_i^{Cl+}$ and fails for $\Rightarrow_i^{As}$. So, the ascriptive counts-as appears to be stronger, but can this be proven semantically? Unfortunately, logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ is not suitable for answering this question, and the reason is the following one.

Proposition 4 *($[u] \neq \Delta_1$). Neither $[u]\varphi \rightarrow \Delta_1\varphi$ nor $\Delta_1\varphi \rightarrow [u]\varphi$ are valid in the class of* $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ *frames.*

Proof *(Sketch).* We provide a countermodel for $\Delta_1\varphi \rightarrow [u]\varphi$. Take a model $\mathcal{M}$ and state w such that $\forall w' \in W$, if $wR_{\mathtt{Iss}}w'$ then $\mathcal{M}, w' \models \varphi$ and $\exists w'' \in W$ such that $wR_u w''$ and $\mathcal{M}, w'' \not\models \varphi$. Model $\mathcal{M}$ is the desired countermodel. An analogous countermodel can be built for $[u]\varphi \rightarrow \Delta_1\varphi$. □

[14] Notice that $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ has the necessary expressive power.

Intuitively, the equivalence fails because R_u and R_1—which, recall, are both equivalence relations—can cross one another's clusters. This is an interesting property of $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ indicating that its model-theory is too weak to force a coincidence between the universal context operator $[u]$ and the full release operator Δ_1 which sounds intuitive, since both $[u]$ and Δ_1 should behave like universal operators.[15] This suggests that logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ should be extended with axiom $[u]\varphi \leftrightarrow \Delta_1\varphi$ to provide a suitable framework for the comparison of the ascriptive view of counts-as with the forms studied in [20,19,18,21,15].

The study of the meta-logical properties of logic $\mathbf{Cxt^u} \otimes \mathbf{PRL} \cup \{[u]\varphi \leftrightarrow \Delta_1\varphi\}$ is left for future work. Now, what is important to be shown is that once the equivalence of $[u]$ and Δ_1 is assumed, the following can be easily proven to hold.

Theorem 3 *($\Rightarrow_i^{As}$ vs. $\Rightarrow_i^{cl+}$). In all $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ frames validating $[u]\varphi \leftrightarrow \Delta_1\varphi$, ascriptive counts-as is strictly stronger than proper classificatory counts-as, that is:*

- *The following formula is valid:*

$$(\varphi_1 \Rightarrow_i^{As} \varphi_2) \rightarrow (\varphi_1 \Rightarrow_i^{cl+} \varphi_2) \tag{30}$$

- *The following formula is not valid:*

$$(\varphi_1 \Rightarrow_i^{cl+} \varphi_2) \rightarrow (\varphi_1 \Rightarrow_i^{As} \varphi_2) \tag{31}$$

Proof. The validity of Formula (30) follows from the fact that in $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ frames $\langle u\rangle\varphi_1 \wedge \neg\varphi_2$ follows from $\neg[i]\Delta_{\sigma(\varphi_2)}(\varphi_1 \rightarrow \varphi_2)$. To prove this, for the semantics of $[i]$ and Δ_I we have that for any model $\mathcal{M}$ and state w, if $\mathcal{M}, w \models \neg[i]\Delta_{\sigma(\varphi_2)}(\varphi_1 \rightarrow \varphi_2)$ then there exists $w \in W_i$ and $w' \in W$ such that $\mathcal{M}, w \models \nabla_1\neg(\varphi_1 \rightarrow \varphi_2)$. Therefore, by $[u]\varphi \leftrightarrow \Delta_1\varphi$, it follows that $\mathcal{M}, w \models \langle u\rangle\neg(\varphi_1 \rightarrow \varphi_2)$. The countermodel for Formula (31) is given by setting $W_i = \emptyset$ and imposing the existence of a world s verifying $\varphi_1 \wedge \neg\varphi_2$. $\square$

The fact that ascriptive counts-as is strictly stronger than proper classificatory lies ultimately on the fact that ascriptive counts-as requires non-empty contexts: $[i]\bot \rightarrow \neg(\varphi_1 \Rightarrow_i^{As} \varphi_2)$. The validity of this property with respect to $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ frames is easily checked semantically. In fact, none of the senses of counts-as analyzed in [20,19,18,21,15] enjoys this property, and in a way this is not surprising, since the ascription of a property to something should presuppose the existence of that something.

4.4. On the properties of the ascriptive reduction

In the two preceding sections we have analyzed the structural properties of ascriptive counts-as. Here we do the same with respect to the ascriptive reduction of deontic notions given in Definition 5. The question we answer with the following two theorems is how the ascriptive reduction of the deontic operators behave with respect to SDL

Theorem 4 *(Validities of deontic ascription). Given Definition 5, the following formulae/rules of $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ are valid/sound in $\mathbf{Cxt^u} \otimes \mathbf{PRL}$:*

$$(\mathsf{O}_i\varphi_1 \wedge \mathsf{O}_i\varphi_2) \rightarrow \mathsf{O}_i(\varphi_1 \wedge \varphi_2) \tag{32}$$

$$\neg\mathsf{O}_i\top \tag{33}$$

$$\vdash \varphi \text{ THEN } \vdash \mathsf{O}_i\varphi \tag{34}$$

Proof. [Formula (32)] We reason semantically proceeding per absurdum. Assume there exists a model $\mathcal{M}$ and state w such that $\mathcal{M}, w \not\models \mathsf{O}_i(\varphi_1 \wedge \varphi_2)$ and $\mathcal{M}, w \models \mathsf{O}_i\varphi_1 \wedge \mathsf{O}\varphi_2$. By Definition 5 we have that $\mathcal{M}, w \models \neg(\neg(\varphi_1 \wedge \varphi_2) \Rightarrow_i^{As} \mathbb{V})$ and hence, by Definition 5, that $\mathcal{M}, w \models \langle i\rangle((\neg\varphi_1 \vee \varphi_2) \wedge \neg\mathbb{V})$ or $\mathcal{M}, w \models [i]\Delta_{\mathbb{V}}((\neg\varphi_1 \vee \neg\varphi_2) \rightarrow \mathbb{V})$. Now, given the assumption, the first disjunct is clearly impossible as $\mathcal{M}, w \models [i]((\neg\varphi_1 \rightarrow \mathbb{V}) \wedge (\neg\varphi_2 \rightarrow \mathbb{V}))$, and the same holds for the second disjuncts as $\mathcal{M}, w \models \neg[i]\Delta_{\mathbb{V}}(\neg\varphi_1 \rightarrow \mathbb{V})$. [Formula (33)] We proceed per absurdum. By Formula (19) and Definition 5 we have, for a pointed model $\mathcal{M}, w$, that $\mathcal{M}, w \models [i](\bot \rightarrow \mathbb{V}) \wedge \langle i\rangle\nabla_{\{\mathbb{V}\}}(\bot \wedge \mathbb{V})$. But the second conjunct is clearly impossible. [Formula (34)] Straightforward by propositional principles. $\square$

Theorem 5 *(Invalidities of deontic ascription). Given Definition 5, the following formulae of $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_m^{Prl}$ are not valid in $\mathbf{Cxt^u} \otimes \mathbf{PRL}$:*

$$\mathsf{O}_i\bot \tag{35}$$

$$\neg\mathsf{O}_i\bot \tag{36}$$

[15] See Section 3.4.

$$\mathsf{O}_i\varphi \rightarrow \neg\mathsf{O}_i\neg\varphi \tag{37}$$

$$\mathsf{O}_i(\varphi_1 \wedge \varphi_2) \rightarrow (\mathsf{O}_i\varphi_1 \wedge \mathsf{O}_i\varphi_2) \tag{38}$$

$$\mathsf{O}_i\varphi_1 \rightarrow \mathsf{O}_i(\varphi_1 \vee \varphi_2) \tag{39}$$

Proof. We provide counter-models for each formula. Let $\varphi = \varphi_1 = p$ and $\varphi_2 = q$ [Formula (35)] By Formula (19) and Definition 5 the desired counter-model can be obtained by setting $\mathcal{I}(\mathrm{V}) = W$. Hence there exists no state w which has access via $\sim_{\mathrm{V}}$ to a state satisfying $\neg\mathrm{V}$. [Formula (36)] To obtain a counter-model it suffices to set $W_i \subseteq \mathcal{I}(\mathrm{V})$ and to require that there exists at least one state falsifying V. [Formula (37)] The counter-model is similar to the one of Formula (36) but it requires the reachability via $\sim_{\mathrm{V}}$ of two states from W_i, one satisfying $p \wedge \neg\mathrm{V}$ and the other satisfying $\neg p \wedge \neg\mathrm{V}$. [Formula (38)] The desired counter-model $\mathcal{M}$ is made like this: $W = \{w_1, w_2\}$, $W_i = \{w_1\}$, $\mathcal{I}(p) = \emptyset$, $\mathcal{I}(q) = \{w_2\}$, $\mathcal{I}(\mathrm{V}) = \{w_1\}$ and $\sim_{\mathrm{V}} = W^2$. We have that $\mathcal{M}, w_1 \models \mathsf{O}_i(p \wedge q) \wedge \neg\mathsf{O}_i q$. [Formula (39)] Let us define the counter-model $\mathcal{M}$ as follows: $W = \{w_1, w_2\}$, $W_i = \{w_1\}$, $\mathcal{I}(p) = \emptyset$, $\mathcal{I}(q) = \{w_2\}$, $\mathcal{I}(\mathrm{V}) = \{w_1\}$ and $\sim_{\mathrm{V}} = W^2$. We have that $\mathcal{M}, w_1 \models \mathsf{O}_i p \wedge \neg\mathsf{O}_i(p \vee q)$. □

Let us comment upon these findings. First of all notice that the necessitation rule (Formula (34)) remains sound. On the other hand, the distribution principle of O_i over $\wedge$ holds only in its right-to-left direction (Formula (32)) while it fails from left to right (Formula (33)). Trivial obligations (Formula (33)) are not possible, and the impossibility of contradictory obligations, as well as their possibility, are both invalid. In particular, by looking at the counter-model of Formula (36) you can notice that $\perp$ is obliged when all states in i are violation states but there is a state which is violation-free and which is reachable via release. An instance of the D axiom of SDL also fails (Formula (37)). Notice that since we do not have distribution over conjunction Formulae (36) and (38) are not equivalent. Perhaps not surprisingly, what also fails is the scheme of the so-called Ross's paradox (Formula (39)): "if you ought to slip the letter into the letter-box, then you ought to slip it or burn it" [36]. The counter-model shows very well why. Although there might be states, indifferent with respect to V, where not sending the letter does not imply a violation, this does not guarantee there also be states where not sending the letter and burning it does not imply a violation. In general, ascribing a property to a set of states does not imply the ascription of the same property to a smaller set of states.

All in all, the definition of deontic operators based on ascriptive counts-as yields a weaker system of SDL. By inspecting the proof of Theorem 5 we can also find what precisely determines the weakness of the logic of O_i as defined in Definition 5, namely the requirement that for $\neg\varphi \Rightarrow_i^{As} \mathrm{V}$ to be true there needs to exist a world, outside context i, which verifies $\neg\varphi$ and $\neg\mathrm{V}$.[16]

4.5. $\mathbf{Cxt^u} \otimes \mathbf{PRL}$, *enthymemes and friendliness*

Before moving on to the next section, in which logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ will be applied to provide a formal analysis of two benchmark problems in deontic logic, we want to put the logic in perspective with some related work done by Makinson in the field of non-monotonic reasoning like, in particular, [32,33].

In Section 2.2 we have briefly recapitulated the notion of contextual classification first introduced in [18]. Contextual classification has been defined as an implicative statement holding with respect to a context, i.e., a set of valuations or, in the modal logic terminology, possible worlds or states (Formula (5)). The role of a context is to limit the set of states with respect to which the implicative statement is evaluated in order for it to represent a classification holding locally in the model. Contexts can therefore be viewed as hidden collections of premises.

Inferences with hidden premises have a long history in logic. The ancient Greeks used to call them *enthymemes* from *en*, in, and *thymos*, mind, so as to mean some knowledge that is left implicit and kept *in the mind*. Contextual classifications state enthymemes in a very precise sense, the hidden premises being the logical theory of their context. A statement "φ_1 counts as φ_2 in context i", interpreted as contextual classification, can therefore be rephrased as "it follows (classically) in the set of propositional models W_i that φ_1 implies φ_2". Enthymemes have been studied as special consequence operations in [32], where they are shown to provide a bridge between classical logic and non-monotonic logics. In that work the notion of enthymeme is captured by a specific logical consequence operation called *pivotal-valuation consequence* [32, Ch. 3]. With slight abuse of notation we denote with $w(\varphi) = 1$ that a propositional valuation w assigns 1 to a propositional formula φ.

Definition 6 *(Pivotal-valuation consequence).* Let W be the set of valuations of a propositional language $\mathcal{L}$ built on $\mathbf{P}$. A formula γ_2 follows from γ_1 modulo the set of valuations $W_i \subseteq W$ iff there is no valuation w in W_i such that $w(\gamma_1) = 1$ and $w(\gamma_2) = 0$.

It is easy to see that Formula (5) is just the formalization of Definition 6 in a modal language. In fact, we can restate Definition 6 in terms of the validity of $\mathbf{Cxt^u}$ formulae on logically universal Cxt models, that is, those models containing

[16] It might be worth noticing that this aspect is strictly related with what discussed in relation to Theorem 3.

all possible valuations of a propositional language $\mathcal{L}$ and all possible contexts on that domain. Let $\mathcal{M}$ be such a model. A propositional formula γ_2 follows from γ_1 modulo the set of valuations $W_i \subseteq W$ iff:

$$\mathcal{M} \models [i](\gamma_1 \to \gamma_2) \quad \big(\text{i.e., } \mathcal{M} \models \gamma_1 \Rightarrow_i^{cl} \gamma_2\big) \tag{40}$$

So contextual classification and pivotal-valuation consequence are, formally speaking, the same notion (in the class of logically universal models). It follows that modal logics at least as strong as $\mathbf{Cxt^u}$ can represent the notion of pivotal-valuation consequence in a modal language.[17]

While pivotal consequence can be expressed already in $\mathbf{Cxt^u}$, the related notion of friendliness [33] requires the fusion $\mathbf{Cxt^u} \otimes \mathbf{PRL}$. Let us first define it.

Definition 7 *(Friendliness).* Let W be the set of valuations of a propositional language $\mathcal{L}$ built on $\mathbf{P}$. A propositional formula γ is friendly for the finite set of formulae Γ if and only if for all valuations w on $\mathbf{P}$ such that $w(\bigwedge \Gamma) = 1$, there exists a valuation w' which agrees with w on all atoms in $\sigma(\Gamma)$ (i.e., the signature of Γ) and such that $w'(\gamma) = 1$.

Also in this case, it is easy to reformulate Definition 7 in language $\mathcal{L}_n^{Cxt} \otimes \mathcal{L}_n^{Prl}$ within the class of logically universal $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ models where the release relation is interpreted as sublanguage equivalence. Let $\mathcal{M}$ be such a model. A propositional formula γ is friendly for the finite set of formulae Γ if and only if

$$\mathcal{M} \models [u]\Big(\bigwedge \Gamma \to \nabla_{-\sigma(\bigwedge \Gamma)}\gamma\Big) \tag{41}$$

In words, all states are such that if they satisfy all the formulae in Γ then, by releasing the whole alphabet except the signature of Γ (i.e., by not releasing the signature of Γ), they have access to a state which satisfies γ. Notice that, unlike the pivotal-consequence relation, friendliness is not monotonic. To appreciate this, set $\Gamma := \{p\}$ and $\gamma = p \wedge q$. Obviously friendliness is lost adding $\neg q$ to Γ.

To conclude, although developed for radically different purposes, logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ features the expressive means to capture two notions of supraclassical consequence relations, one monotonic (contextual classification), one not (friendliness). Although of definite interest, a detailed investigation of the formalization of these notions in $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ goes beyond the scope of the present paper. So, in the next section, we turn to the applications of ascriptive counts-as in the context of deontic logic.

5. Putting the ascriptive view at work

Section 4.4 has put the ascriptive view of norms, as formalized here, in perspective with standard deontic logic showing also how, for instance, it does not suffer of Ross's paradox. In the present section we proceed at situating the ascriptive view of norms with respect to some classical issues of deontic logic. In particular we show how this view of norms offers an original perspective from which to address Chisholm's paradox [11] and Jørgensen's dilemma [25].

5.1. An ascriptive glance at Chisholm's paradox

We follow the presentation of the paradox given in [5].

1. It ought to be that Smith refrains from robbing Jones.
2. Smith robs Jones.
3. If Smith robs Jones, he ought to be punished for robbery.
4. It ought to be that if Smith refrains from robbing Jones he is not punished for robbery.

Once "Smith robbing Jones" is represented by r and "Smith refraining from robbing Jones" by $\neg r$ and, similarly, "Smith being punished" by p while "Smith not being punished" by $\neg p$, this set of ordinary language sentences—also called the Chisholm's set—can receive the following formalizations within SDL, which differ in the way they symbolize the conditional statements at points 3 and 4.

i)		ii)		iii)		iv)	
	$\mathsf{O}\neg r$		$\mathsf{O}\neg r$		$\mathsf{O}\neg r$		$\mathsf{O}\neg r$
	r		r		r		r
	$r \to \mathsf{O}p$		$\mathsf{O}(r \to p)$		$r \to \mathsf{O}p$		$\mathsf{O}(r \to p)$
	$\neg r \to \mathsf{O}\neg p$		$\mathsf{O}(\neg r \to \neg p)$		$\mathsf{O}(\neg r \to \neg p)$		$\neg r \to \mathsf{O}\neg p$

[17] The interested reader is referred to [18] where it is shown that the operator $\Rightarrow_c^{cl}$ of contextual classification satisfies all the properties characterizing pivotal-valuation consequences, that is: reflexivity, monotonicity, transitivity, disjunction of premises, and supraclassicality, where supraclassicality means that via pivotal-valuation consequence more can be inferred than what can be classically inferred, which is what is guaranteed by axiom schema `Incl` in logic $\mathbf{Cxt^u}$.

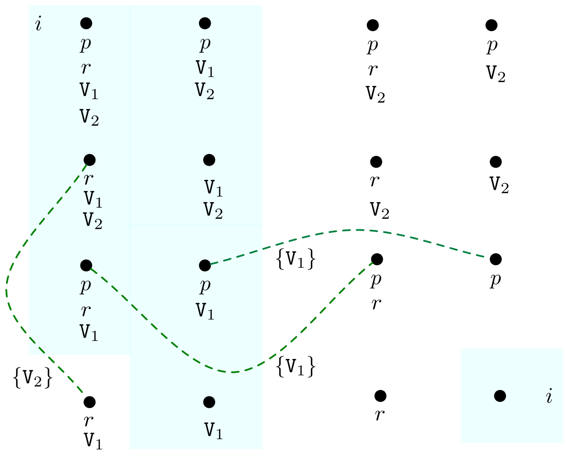

Fig. 2. $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ model of the Chisholm's scenario.

It becomes then evident that SDL falls short in properly representing the Chisholm's set since: in i) the 4th statement $\neg r \rightarrow \mathsf{O}\neg p$ is a logical consequence of the 2nd r, which is not the case in the ordinary language formulation; in ii) the 3rd statement $\mathsf{O}(r \rightarrow p)$ is a logical consequence of the 1st $\mathsf{O}\neg r$, which is also not the case in the ordinary language formulation; finally, in iii) and iv) both $\mathsf{O}p$ and $\mathsf{O}\neg p$ are logical consequences of the set and hence $\mathsf{O}\bot$, which is clearly counter-intuitive.

Let us see how the Chisholm's set can be dealt with in our framework. The better way to do that is to have a model-theoretic look at the paradox. Let $\mathbf{P} := \{p, r, \mathbb{V}_1, \mathbb{V}_2\}$ and $\mathbb{C} = \{i\}$.[18] Notice that, unlike in the Adam & Eve's example, two violation constants are needed here: $\mathbb{V}_1$ for the ascription of primary obligations, and $\mathbb{V}_2$ for the ascription of contrary-to-duty obligations. The $\mathsf{TE}^{eq} \otimes \mathsf{Prl}$ model $\mathcal{M} = \langle W, \{W, W_i\}, \{\sim_I\}_{I \in 2^{\mathbf{P}}}, \mathcal{I}\rangle$ representing the Chisholm's scenario is given in Fig. 2. The model consists of all possible propositional valuations of $\mathbf{P}$, some of which are clustered in context i (the states belonging to the context are highlighted). Such context consists of the truth set of the propositional version of the first three sentences of the Chisholm's set, i.e., $\{r \rightarrow \mathbb{V}_1, (r \wedge \neg p) \rightarrow \mathbb{V}_2, p \rightarrow \mathbb{V}_1\}$.[19] To reduce clutter, only the relevant sub-language equivalence relations are depicted. It is easy to see that the model validates the following representation of the first three sentences of the Chisholm's set in terms of ascriptive counts-as statements:

$$r \Rightarrow_i^{As} \mathbb{V}_1 \tag{42}$$

$$(r \wedge \neg p) \Rightarrow_i^{As} \mathbb{V}_2 \tag{43}$$

$$p \Rightarrow_i^{As} \mathbb{V}_1 \tag{44}$$

In addition, it also validates $[i](\mathbb{V}_2 \rightarrow \mathbb{V}_1)$ and $\neg[i](\mathbb{V}_1 \rightarrow \mathbb{V}_2)$. That is, the occurrence of a violation of a contrary-to-duty norm logically implies that a violation of a primary norm already occurred, and not vice versa.[20] Intuitively, this is so because the violation of a contrary-to-duty norm cannot occur without a violation of a primary norm.

Given this representation, it is clear that nothing paradoxical happens in states satisfying r (i.e., the last sentence of the Chisholm's set). In those states, which are $\mathbb{V}_1$-states, if p is not the case, than also $\mathbb{V}_2$ is the case. To get back to ordinary language, Fig. 2 shows that a natural and consistent interpretation of the Chisholm's scenario in terms of ascriptions goes as follows:

1. A primary violation is ascribed to states in which Smith robs Jones (Formula (42));
2. Smith robs Jones;
3. A contrary-to-duty violation is ascribed to states in which Smith robs Jones—and hence in which a primary violation already occurred—and he is not punished (Formula (43));
4. A primary violation is ascribed to states in which Smith is punished (Formula (44)).

The Chisholm's scenario can therefore be accommodated in a rather natural way by means of ascriptive counts-as and assuming two violation atoms logically related in such a way that the worse violation results to be logically stronger. With respect to this, notice also that in case we want to introduce deontic operators by definitions of the type given

[18] Only one context is at issue in the paradox.

[19] This context might seem to include too many states (e.g., the state satisfying only $\mathbb{V}_1$). A smaller context could be obtained by replacing implications with biimplications. We refrain from doing so as we want to stick to the letter of the paradox.

[20] These two last constraints could also be set to be of a global type, that is, to occur in the scope of $[u]$ instead of $[i]$.

in Definition 5, such operators will need to be indexed with the violation to which they pertain, e.g., $\mathsf{O}_i^{\mathtt{V}_1}$ and $\mathsf{O}_i^{\mathtt{V}_2}$. So, Formulae (42)–(44) could be rewritten as:

$$\mathsf{O}_i^{\mathtt{V}_1}\neg r \tag{45}$$

$$\mathsf{O}_i^{\mathtt{V}_2}(r \to p) \tag{46}$$

$$\mathsf{O}_i^{\mathtt{V}_1}\neg p \tag{47}$$

where each obligation keeps track of the ascription on which it depends.

5.2. An ascriptive glance at Jørgensen's dilemma

Before concluding, we now show how Definition 5 provides an interesting angle from which to look at Jørgensen's dilemma [25]. The first of the ten philosophical problems urging today's deontic logic according to [22] was the problem, already formulated in [31], concerning a suitable foundation of deontic logic in the face of Jørgensen's dilemma:

> "How can deontic logic be reconstructed in accord with the philosophical position that norms are neither true nor false?" [22, p. 3]

It is our claim that the ascriptive view of norms can provide the ground for such a reconstruction. Let us sketch how this would work in the case of Adam & Eve scenario. There, God's commandment does three different things at the same time. First, the commandment defines the context i of divine norms. As such, formula $\mathtt{eat\_apple} \to \mathtt{V}$ defines the "logical space" [35, p. 6] of the normative system at issue, i.e., its context (e.g., states w_1, w_3, w_4 in Adam & Eve's example). Notice that, as such, $\mathtt{eat\_apple} \to \mathtt{V}$ is properly speaking neither true nor false, but it is rather taken or assumed to be true, exactly like an axiom. Second, the commandment teaches Adam & Eve how to recognize, to say it with Searle [38], states with a certain "institutional" property ('violation') on the ground of a "brute" property ('eating the apple'). Third, the commandment increases the granularity of Adam & Eve's language so that they can distinguish state w_1 from state w_2 (and w_3 from w_4) by making use of suitable "institutional" terms. This is the aspect of language creation proper of the ascriptive view of norms. To sum up, a norm $\varphi \to \mathtt{V}$ in a set of norms Φ works like an axiom defining the context i of the set of norms Φ, and defines the violation term $\mathtt{V}$ by ascribing it to term φ built from some "brute" language.

With respect to the third point, notice that the statement $\mathtt{eat\_apple} \to \mathtt{V}$ is neither true nor false if a "brute" language is spoken, where the "institutional" term $\mathtt{V}$ is not used. In fact, in the scenario there are states in the model where neither $\Delta_{\{\mathtt{V}\}}(\mathtt{eat\_apple} \to \mathtt{V})$ nor $\Delta_{\{\mathtt{V}\}}\neg(\mathtt{eat\_apple} \to \mathtt{V})$ are true. That is why, to say it with Jørgensen, norms correspond to "quasi-properties" of reality [25, pp. 292–293]. Properties, or to use Searle's terminology again, "brute facts" hold independently of the human ascriptive activity, while "quasi-properties" or "institutional facts" hold only as a result of ascription, and in this sense they are in a way less true. Notice, however, that this notion of truth is not the technical one used in Kripke semantics: the notion of truth in Jørgensen's dilemma (i.e., truth as what is evaluated as true given the brute language) is not the Kripke notion of truth (i.e., truth as what is evaluated as true given the whole language). The logic presented here generalizes this distinction to any possible partition besides the "brute" vs. "institutional" one.

6. Conclusions and future work

The paper has analyzed, by making use of modal logic techniques, the notion of ascription as it emerges from some philosophical contributions to the theory of norms. In particular, by providing Anderson's reduction with sufficient modal means for supporting a notion of context and of linguistic indistinguishability, the paper has provided an original definition of deontic statements as forms of ascriptions (Definition 5). Such definition has been compared with SDL (Theorems 4 and 5) and its underlying notion of counts-as has also been confronted with formalizations of counts-as available in the literature (Theorems 1–3). The proposal has then been applied to provide original analyses of the Chisholm's paradox and of Jørgensen's dilemma.

Future work will focus on two aspects. First, the study of logic $\mathbf{Cxt^u} \otimes \mathbf{PRL}$ extended with axiom $[u]\varphi \leftrightarrow \Delta_1\varphi$, and of the less standard fusion of logic $\mathbf{Cxt^u}$ with the logic characterizing equivalence up to a signature presented in [16]. This will give a full spectrum of modal systems offering means for the clarification of the notion of ascription. Second, all the systems taken into consideration in the paper are of a static kind. The dynamic aspect of ascription—including both aspects of context and release dynamics—will be studied by making use of some form of update logic in the spirit of, for instance, [8]. Some results in that direction have already been published in [7].

Acknowledgements

This study was partly supported by the *Ministère de la Culture, de L'Enseignement Supérieur et de la Recherche, Grand-Duché de Luxembourg* (grant BFR07/123) and partly by the *Nederlandse Organisatie voor Wetenschappelijk Onderzoek* under the VENI grant Nr. 639.21.816. The author is very grateful to the anonymous referees of DEON'08, to the anonymous referees of the

Journal of Applied Logic, to Prof. Leendert van der Torre, Prof. Guido Boella and Paolo Turrini. Their accurate comments helped me in considerably improving the paper, in both its content and exposition.

Appendix A. Alternative semantics for $\mathbf{Cxt^u}$

The present section introduces an alternative semantics for language $\mathcal{L}_n^{Cxt}$ based on a class of frames whose accessibility relations are able to simulate contexts. Such class of frames is proven equivalent to the class of Cxt frames introduced in Section 2.3.

Language $\mathcal{L}_n^{Cxt}$ can be given a semantics via the class of TE^{eq} frames $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathrm{C}}\rangle$ such that:

- W is a non-empty set of states;
- $\{R_i\}_{i\in\mathrm{C}}$ is a family of relations which are i–j transitive (if wR_iw' and $w'R_jw''$ then wR_jw''), i–j euclidean (if wR_iw' and wR_jw'' then $w'R_jw''$)[21];
- there exists a relation $R_u \in \{R_i\}_{i\in\mathrm{C}}$ such that R_u is an equivalence relation and for all $i \in \mathrm{C}, w \in W$: wR_iw' implies wR_uw_i.

In short, these frames consist of the domain W and of a finite number of accessibility relations containing an equivalence relation R_u such that, for all $i \in \mathrm{C}$, $R_i \subseteq R_u$. The following proposition clarifies the relationship between this semantic, and the one based on Cxt frames.

Proposition 5 *(Semantic equivalence of* Cxt *and* TE^{eq} *frames). For any formula* φ *on* $\mathcal{L}_n^{Cxt}$: $\mathsf{TE}^{eq} \models \varphi$ *iff* $\mathsf{Cxt} \models \varphi$*. That is,* Cxt *frames and* TE^{eq} *frames define the same logic.*

Proof. [$\Leftarrow$] From right to left: for every φ, $\textsc{Cxt} \models \varphi$ implies $\mathsf{TE}^{eq} \models \varphi$. The proof is obtained by showing that a $\textsc{Cxt}$ can always be transformed into a TE^{eq} frame in a way that preserves modal formulae. $\mathcal{F} = \langle W, \{W_i\}_{i\in\mathrm{C}}\rangle$ be a Cxt frame. Define now for all $i \in \mathrm{C}$ a relation $R_i \subseteq W^2$ such that wR_iw' iff $w' \in W_i$. Clearly, R_u is the universal relation and every universal relation is an equivalence relation, which also includes all R_i's for any $i \in \mathrm{C}$. Now, to prove i–j transitivity, suppose that wR_iw' ($w' \in W_i$) and $w'R_jw''$ ($w'' \in W_j$). It follows therefore that wR_jw''. The proof of i–j euclidicity is perfectly analogous. Suppose that wR_iw' ($w' \in W_i$) and wR_jw'' ($w'' \in W_j$), hence $w'R_jw''$. Hence, the constructed frame $\langle W, \{W_i\}_{i\in\mathrm{C}}\rangle$ is a TE^{eq} frame. It follows that for any Cxt model a TE^{eq} model can be built by simply keeping the same valuation function which clearly satisfies the same $\mathcal{L}_n^{Cxt}$ formulae. [$\Rightarrow$] From left to right: for every φ, $\mathsf{TE}^{eq} \models \varphi$ implies $\textsc{Cxt} \models \varphi$. In this case, the proof is obtained by showing that every TE^{eq} frame, which is also point-generated, can be turned into a context frame which is equivalent with respect to $\mathcal{L}_n^{Cxt}$ formulae. Let us denote with $g(\mathsf{TE}^{eq})$ the class of point-generated TE^{eq} frames.[22] From modal logic we know that point-generated subframes preserve modal validity [9, Ch. 3.3], so it suffices to show that every $g(\mathsf{TE}^{eq})$ frame can be transformed into a $\textsc{Cxt}$ frame. For any R_i let $r_i(w) = \{w' \mid wR_iw'\}$. The desired construction is obtained by setting, for any $i \in \mathrm{C}$, $W_i = r_i(w)$ where w is the root of the point-generated frame. It can now be shown that for every w', w'' if there exists an R_i-path from w to w' and from w to w'', then $w'R_iw''$ iff $w'' \in r_i(w)$. From left to right, if there exists an R_i-path from w to w' and $w'R_iw''$, then by transitivity (which is a special case of i–j transitivity) wR_iw'', that is, $w'' \in r_i(w)$. From right to left, if there exists an R_i-path from w to w' and $w'' \in r_i(w)$, then wR_iw'' and hence, by euclidicity, $w'R_iw''$. It is then easy to see that $r_u(w) = W$. □

So TE^{eq} frames are modally equivalent, with respect to language $\mathcal{L}_n^{Cxt}$, to Cxt frames. However, as made explicit by the proof, they consist of a larger class of structures, as Cxt frames are in one-to-one correspondence only with the class of point-generated TE^{eq} frames. Although this difference cannot be noticed by $\mathcal{L}_n^{Cxt}$ formulae, it does have an important model-theoretical consequence, namely that while the class of Cxt frames is not closed under taking disjoint unions,[23] the class of TE^{eq} frames is. This depends, ultimately, on the fact that Cxt frames contain a universal relation, which is notoriously not closed under taking disjoint unions, while TE^{eq} frames simply work with an equivalence relation containing all the contextual accessibility relations. This model theoretic properties is essential for the fusion studied in Section 4. This notwithstanding, Cxt frames are clearly more appealing from an intuitive point of view, as they are less technical and able to capture the view contexts as sets of (propositional) models in a direct way.

References

[1] A.R. Anderson, The formal analysis of normative concepts, American Sociological Review 22 (1957) 9–17.
[2] A.R. Anderson, The logic of norms, Logique et Analyse 2 (1958) 84–91.
[3] A.R. Anderson, A reduction of deontic logic to alethic modal logic, Mind 22 (1958) 100–103.

[21] The terminology is borrowed from [29] where a similar class of frames is studied.
[22] Recall that a point-generated frame is a frame in which for each accessibility relation R_i, each state is reachable from the root via an R_i path [9].
[23] For this important model-theoretic notion the reader is referred to [9, Ch. 3.3].

[4] A.R. Anderson, Some nasty problems in the formal logic of ethics, Nôus 1 (1967) 345–360.
[5] L. Åqvist, Good samaritans, contrary-to-duty imperatives and epistemic obligations, Nous 1 (1967) 361–379.
[6] L. Åqvist, Deontic logic, in: D. Gabbay, F. Guenthner (Eds.), Handbook of Philosophical Logic. Vol. II: Extensions of Classical Logic, Reidel, 1984, pp. 605–714.
[7] G. Aucher, D. Grossi, A. Herzig, E. Lorini, Dynamic context logic, in: X. He, J. Horty, E. Pacuit (Eds.), Proceedings of LORI 2009, in: LNAI, vol. 5834, Springer, 2009.
[8] J. van Benthem, J. van Eijck, B. Kooi, Logics of communication and change, Information and Computation 204 (11) (2006) 1620–1662.
[9] P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, Cambridge University Press, Cambridge, 2001.
[10] H. Bohnert, The semiotic status of commands, Philosophy of Science 12 (4) (1945) 302–315.
[11] R.M. Chisholm, Contrary-to-duty imperatives and deontic logic, Analysis 23 (1963) 33–36.
[12] P. d'Altan, J.-J.Ch. Meyer, R.J. Wieringa, An integrated framework for ought-to-be and ought-to-do constraints, in: Y.H. Tan (Ed.), Working Papers of the Workshop om Deontic and Non-Monotonic Logics, Rotterdam, 1993, pp. 14–45.
[13] D.M. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev, Many-Dimensional Modal Logics. Theory and Applications, Elsevier, 2003.
[14] C. Ghidini, F. Giunchiglia, Local models semantics, or contextual reasoning = locality + compatibility, Artificial Intelligence 127 (2) (2001) 221–259.
[15] D. Grossi, Designing invisible handcuffs. Formal investigations in institutions and organizations for multi-agent systems, PhD thesis, Utrecht University, SIKS, 2007.
[16] D. Grossi, Linguistic relevance in modal logic, in: A. Nijholt, M. Pantic, M. Poel, G.H.W. Hondorp (Eds.), Proceedings of the 20th Belgian–Netherlands Conference on Artificial Intelligence (BNAIC'08), University of Twente, 2008.
[17] D. Grossi, A note on brute vs. institutional facts: Modal logic of equivalence up to a signature, in: Guido Boella, Pablo Noriega, Gabriella Pigozzi, Harko Verhagen (Eds.), Normative Multi-Agent Systems, in: Dagstuhl Seminar Proceedings, vol. 09121, Dagstuhl, Germany, 2009.
[18] D. Grossi, J.-J.Ch. Meyer, F. Dignum, Classificatory aspects of counts-as: An analysis in modal logic, Journal of Logic and Computation 16 (5) (2006) 613–643, Oxford University Press.
[19] D. Grossi, J.-J.Ch. Meyer, F. Dignum, Counts-as: Classification or constitution? An answer using modal logic, in: L. Goble, J.-J.Ch. Meyer (Eds.), Proceedings of DEON 2006, in: LNAI, vol. 4048, 2006, pp. 115–130.
[20] D. Grossi, J.-J.Ch. Meyer, F. Dignum, Modal logic investigations in the semantics of counts-as, in: Proceedings of the Tenth International Conference on Artificial Intelligence and Law (ICAIL'05), ACM, June 2005, pp. 1–9.
[21] D. Grossi, J.-J.Ch. Meyer, F. Dignum, The many faces of counts-as: A formal analysis of constitutive-rules, Journal of Applied Logic 6 (2) (2008) 192–217.
[22] Jörg Hansen, Gabriella Pigozzi, and Leendert van der Torre. Ten philosophical problems in deontic logic, in: Guido Boella, Leon van der Torre, Harko Verhagen (Eds.), Normative Multi-Agent Systems, No. 07122 in DROPS Proceedings, Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2007.
[23] H.L.A. Hart, The ascription of responsibility and rights, in: A. Flew (Ed.), Logic and Language, Blackwell, 1951.
[24] A.J.I. Jones, M. Sergot, A formal characterization of institutionalised power, Journal of the IGPL 3 (1996) 427–443.
[25] J. Jørgensen, Imperatives and logic, Erkentniss (1937) 288–296.
[26] S. Kanger, New fondations for ethical theory, in: R. Hilpinen (Ed.), Deontic Logic: Introductory and Systematic Readings, Reidel Publishing Company, 1971, pp. 36–58.
[27] J. Krabbendam, J.-J.Ch. Meyer, Release logics for temporalizing dynamic logic, orthogonalising modal logics, in: M. Barringer, M. Fisher, D. Gabbay, G. Gough (Eds.), Advances in Temporal Logic, Kluwer Academic Publisher, 2000, pp. 21–45.
[28] J. Krabbendam, J.-J.Ch. Meyer, Contextual deontic logics, in: P. McNamara, H. Prakken (Eds.), Norms, Logics and Information Systems, IOS Press, Amsterdam, 2003, pp. 347–362.
[29] A. Lomuscio, M. Sergot, Deontic interpreted systems, Studia Logica 75 (2003) 63–92.
[30] E. Lorini, D. Longin, A logical account of institutions: from acceptances to norms via legislators, in: G. Brewka, J. Lang (Eds.), International Conference on Principles of Knowledge Representation and Reasoning (KR), Sidney, Australia, 16/09/08–19/09/08, AAAI Press, 2008, pp. 38–48.
[31] D. Makinson, On a fundamental problem of deontic logic, in: P. McNamara, H. Prakken (Eds.), Norms, Logics and Information Systems. New Studies in Deontic Logic and Computer Science, in: Frontiers in Artificial Intelligence and Applications, vol. 49, IOS Press, Amsterdam, 1999, pp. 29–53.
[32] D. Makinson, Bridges from Classical to Nonmonotonic Logic, Texts in Computing, vol. 5, King's College Publications, 2005.
[33] D. Makinson, Friendliness and sympathy in logic, in: Logica Universalis, Birkhäuser Verlag, 2005, pp. 191–206.
[34] S. Pufendorf, De Jure Naturae et Gentium, Clarendon Press, 1934 (edition of 1688).
[35] M. Ricciardi, Constitutive rules and institutions, Paper presented at the meeting of the Irish Philosophical Club, Ballymascanlon, February 1997.
[36] A. Ross, Imperatives and logic, Philosophy of Science 11 (1) (1944) 30–46.
[37] J. Searle, Speech Acts. An Essay in the Philosophy of Language, Cambridge University Press, Cambridge, 1969.
[38] J. Searle, The Construction of Social Reality, Free Press, 1995.
[39] R. Stalnaker, On the representation of context, Journal of Logic, Language, and Information 7 (1998) 3–19, Kluwer.